J. Appl. Maths Mechs Vol. 56, No. 5, pp. 665–673, 1992
Printed in Great Britain.

0021–8928/92 $24.00 + .00

THE NON-AXISYMMETRIC PROBLEM OF THE STRESS CONCENTRATION IN AN UNBOUNDED ELASTIC MEDIUM NEAR A SPHERICAL SLIT†

G. YA. POPOV

Odessa

(*Received* 24 *June* 1991)

Using the approach described in [1], the non-axisymmetric problem of the stress concentration in an unbounded elastic medium near a spherical slit is reduced to three, one-dimensional, separately solvable, integral equations. Their exact solution is obtained in a class of functions with non-integrable singularities and is used, as in [1], to derive simple formulae for the stress intensity factors. The axisymmetric modification of this problem has been investigated in [2–5].

1. MODIFICATION OF THE TREFFTZ REPRESENTATION

WE ARE interested in the representation [6]

$$v = \psi + (r^2 - R^2)\operatorname{grad}\psi_0 \tag{1.1}$$

Here $v = 2Gu$, u is the displacement vector, and ψ is a harmonic vector with components $u_j(x_1, x_2, x_3)$, $\psi_j(x_1, x_2, x_3)$ $(j = 1, 2, 3)$, respectively, and ψ_0 is a harmonic function satisfying the equation

$$r\frac{\partial\psi_0}{\partial r} + \nu\psi_0 = -\frac{\operatorname{div}\psi}{2\kappa}, \quad \nu = \frac{1-2\mu}{\kappa}$$
$$(\kappa = 3 - 4\mu,\ r = \sqrt{x_1^2 + x_2^2 + x_3^2}) \tag{1.2}$$

where μ is Poisson's ratio, and G is the shear modulus.

If we have formula (1.1) for the components of the displacement vector in a Cartesian system of coordinates, well-known formulae can be used to express those components in a spherical system of coordinates and then, using the Cauchy relations and Hooke's law, to find the stresses σ_r, $\tau_{r\theta}$, $\tau_{r\varphi}$. However, these formulae prove to be quite complicated. We will therefore obtain formulae of the same structure as (1.1), as in [1]. To do this, we obtain [7] a formula for the components $p_{\nu j}$ $(j = 1, 2, 3)$ of the stress vector over the cross-section with normal ν. Using (1.1), the Cauchy relations and Hooke's law, together with (1.2) and the equation $\operatorname{div}v = \operatorname{div}\psi + 2r\partial\psi_0/\partial r$, this formula can be written in the form

$$rp_{\nu j} = H_j(x_1, x_2, x_3) + (r^2 - R^2)r\frac{\partial\psi_0(x_1, x_2, x_3)}{\partial r\partial x_j}$$
$$H_j = -2\mu x_j\psi_0 + r^2\frac{\partial\psi_0}{\partial x_j} + x_j r(1-2\mu)\frac{\partial\psi_0}{\partial r} + \frac{1}{2}\sum_{k=1}^{3} x_k\left(\frac{\partial\psi_j}{\partial x_k} - \frac{\partial\psi_k}{\partial x_j}\right),\ j = 1, 2, 3 \tag{1.3}$$

It can be shown [7] that $\Delta H_j = 0$.

† *Prikl. Mat. Mekh.* Vol. 56, No. 5, pp. 770–779, 1992.

Having the formula for $p_{\nu j}$, we find

$$\sigma_r = p_{\nu 1}\cos\varphi\sin\theta + p_{\nu 2}\sin\varphi\sin\theta + p_{\nu 3}\cos\theta$$

$$\tau_{r\theta} = p_{\nu 1}\cos\varphi\cos\theta + p_{\nu 2}\sin\varphi\cos\theta - p_{\nu 3}\sin\theta$$

$$\tau_{r\varphi} = -p_{\nu 1}\sin\varphi + p_{\nu 2}\cos\varphi$$

or, after obvious algebra

$$r\sigma_r = \psi_1^*(r,\theta,\varphi) + (r^2 - R^2)r\psi_0''(r,\theta,\varphi)$$

$$r\tau_{r\theta} = \psi_2^*(r,\theta,\varphi) + (r^2 - R^2)r[r^{-1}\dot{\psi}_0(r,\theta,\varphi)]'$$

$$r\tau_{r\varphi} = \psi_3^*(r,\theta,\varphi) + (r^2 - R^2)r[(r\sin\theta)^{-1}\psi_0'(r,\theta,\varphi)]' \tag{1.4}$$

$$\psi_1^* = H_1\cos\varphi\sin\theta + H_2\sin\varphi\sin\theta + H_3\cos\theta$$

$$\psi_2^* = H_1\cos\varphi\cos\theta + H_2\sin\varphi - H_3\sin\theta$$

$$\psi_3 = -H_1\sin\varphi + H_2\cos\varphi \tag{1.5}$$

(the derivative with respect to r is denoted by the prime, the derivative with respect to θ by the dot, and the derivative with respect to φ by the comma). Then, introducing the function

$$\psi_4^* = H_1\cos\varphi + H_2\sin\varphi$$

and the complex combination

$$\psi_i^* = \psi_4^* + i\psi_3^*$$

it is easy to see that ψ_i^* satisfies the differential equation

$$\Delta\psi_i^* - [\psi_i^* - 2i(\psi_i^*)^{,}]r^{-2}\operatorname{cosec}^2\theta = 0 \tag{1.6}$$

It can also be shown that the function ψ_j^* $(j = 1, 2, 3)$ can be expressed in terms of the solution of Eq. (1.6) by the formulae

$$2\psi_1^* = (\psi_i^* + \overline{\psi_i^*})\sin\theta + H_3\cos\theta$$

$$2\psi_2^* = (\psi_i^* + \overline{\psi_i^*})\cos\theta - H_3\sin\theta \tag{1.7}$$

$$2i\psi_3^* = \psi_i^* - \overline{\psi_i^*}(\Delta H_3 = 0)$$

and after this has been done the harmonic function ψ_0 can be found.

Thus, we introduce vector functions H and ψ^* with components H_1, H_2, H_3 and ψ_1^*, ψ_2^*, ψ_3^*, respectively. Using formula (1.3) for H_j, we calculate the derivatives of the components with respect to x_j. This gives

$$-\operatorname{div} H = 2[r(r\psi_0')' - 2(1-2\mu)r\psi_0' - 2(1+\mu)\psi_0] \tag{1.8}$$

On the other hand, using formula (1.5) for H_j, we establish the equation $\operatorname{div} H = \operatorname{div}\psi^*$, and this, together with (1.8), gives an equation for finding ψ_0.

2. STATEMENT OF THE PROBLEM AND CONSTRUCTION OF A DISCONTINUOUS SOLUTION

We will assume that there is a spherical slit of radius R with centre at the origin of coordinates $r = 0$, $\theta = 0$, $\varphi = 0$ in an unbounded elastic medium with shear modulus G and Poisson's ratio μ. Let the slit occupy part of the sphere: $0 \leqslant \theta \leqslant \omega$, $-\pi \leqslant \varphi \leqslant \pi$. The elastic medium is loaded arbitrarily and the stress distribution when there is no slit is known

$$\sigma_r = -q_1(r,\theta,\varphi),\quad \tau_{r\theta} = -q_2(r,\theta,\varphi),\quad \tau_{r\varphi} = -q_3(r,\theta,\varphi) \tag{2.1}$$

The problem consists of finding the stress distribution in the elastic medium when there is a slit in it and, in particular, in finding the stress intensity factors at the slit edges.

We construct the required stress field in the form

$$\sigma_r = \sigma_r^* - q_1, \qquad \tau_{r\theta} = \tau_{r\theta}^* - q_2, \qquad \tau_{r\varphi} = \tau_{r\varphi}^* - q_3 \tag{2.2}$$

where the stresses indicated by an asterisk are constructed by (1.4) and (1.5), and we allow for discontinuity of the displacement field on crossing the slit. To do so, we must construct discontinuous solutions of Eq. (1.6) and the harmonic equation $\Delta H_3 = 0$ with first-order discontinuities on the slit $r = R$ with prescribed jumps of the required functions and their normal derivatives (normal to the line of discontinuities), i.e.

$$\begin{gathered}[H_3],\ [H_3'],\ [\psi_i^*],\ [\psi_i^{*\prime}]\\ [F] = F(R-0,\theta,\varphi) - F(R+0,\theta,\varphi)\end{gathered} \tag{2.3}$$

These discontinuous solutions are constructed by applying a Fourier transformation with respect to the polar angle to the given equations

$$H_{3n}(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_3(r,\theta,\varphi)e^{-in\varphi}d\varphi, \quad n = 0, \pm 1, \pm 2, \ldots \tag{2.4}$$

and then a Legendre transformation with respect to θ

$$H_{3nk}(r) = \int_0^{\pi} \sin\theta\, P_k^{(n)}(\cos\tau)H_{3n}(r,\theta)d\theta \tag{2.5}$$

and, finally, the Mellin integral transformation of the generalized model [8]. Inverting the resulting Mellin transforms and then the Legendre transform using the formula

$$\begin{gathered}H_{3n}(r,\theta) = \sum_{k=|n|}^{\infty} H_{3nk}(r)\sigma_{k,|n|}P_k^{|n|}(\cos\theta)\\ 2\sigma_{k,m} = (k-m)!\,[(k+m)!]^{-1}(2k+1)\end{gathered} \tag{2.6}$$

we obtain the following formula for the Fourier transforms of the required discontinuous solutions

$$H_n(r,\theta) = R^2\ \{ \int \sin\tau K_n^*(r,R,\theta,\tau)[H_n'(R,\tau)]d\tau - \frac{\partial}{\partial R}\int \sin\tau K_n^*(r,R,\theta,\tau)[H_n(R,\tau)]d\tau \tag{2.7}$$

and a similar expression for $\psi_{in}^*(r,\theta)$, in which K_n^* is replaced by K_{n+1}^*, where

$$\begin{gathered}K_m^*(r,R,\theta,\tau) = \sum_{k=|m|}^{\infty} \sigma_{k,|m|}\Psi_k(r,R)P_k^{|m|}(\cos\theta)P_k^{|m|}(\cos\tau)\\ (2k+1)\Psi_k(r,R) = r^kR^{-k},\quad r<R;\quad (2k+1)\Psi_k(r,R) = R^{k+1}r^{-k-1},\ r>R\end{gathered} \tag{2.8}$$

Here and below the integration with respect to τ is taken from $\tau = 0$ to $\tau = \omega$.

Clearly, in order to find the Fourier transforms of the required stresses σ_r^*, $\tau_{r\theta}^*$, $\tau_{r\varphi}^*$ using (1.4) and (1.5), written in terms of Fourier transforms, we need to find the transforms of the discontinuities of the functions ψ_i^* and their derivatives. We obtain these discontinuities from the condition for zero stresses (2.2) on the sides of the slit $r = R \mp 0$, that is

$$\sigma_r^*|_{r=R\mp 0} = q_1, \quad \tau_{r\theta}^*|_{r=R\mp 0} = q_2, \quad \tau_{r\varphi}^*|_{r=R\mp 0} = q_3$$

which can be written in transforms as

$$\begin{gathered}\sigma_{rn}^*(R\mp 0,\theta) = q_{1n}(\theta),\quad \tau_{\theta n}^*(R\mp 0,\theta) = q_{2n}(\theta),\\ \tau_{\varphi n}'(R\mp 0,\theta) = q_{3n}(\theta),\quad 0\leqslant\theta\leqslant\omega,\quad n = 0,\pm 1,\pm 2,\ldots\end{gathered} \tag{2.9}$$

It follows immediately from (1.4) that $[\psi_{jn}^*(R,0)] = 0$ $(j = 1, 2, 3)$ and thus, from (1.5)

$$[H_n(R, \theta)] = 0, \quad [\psi^*_{in}(R, \theta)] = 0 \tag{2.10}$$

This simplifies (2.7) and the analogous formula for $\psi^*_{in}(r, \theta)$ considerably

$$H_n(r, \theta) = R^2 \int \sin\tau K^*_n(r, R, \theta, \tau)[H'_n(R, \tau)]\, d\tau \tag{2.11}$$
$$\psi^*_{in}(r, \theta) = R^2 \int \sin\tau K^*_{n+1}(r, R, \theta, \tau)[\psi^{*\prime}_{in}(R, \tau)]\, d\tau$$

3. REDUCTION OF THE PROBLEM IN QUESTION TO AN INTEGRAL EQUATION

From Sec. 2, it is clear that, in order to find H_3 and ψ^*_{in}, and then ψ^*_j ($j = 1, 2, 3$) from (1.7), it is sufficient to find the jumps $[H_n'(R, \tau)]$ and $[\psi^{*\prime}_{in}(R, \tau)]$. For this, conditions (2.9) must be satisfied, and from (1.4) and (2.10) these reduce to the equations

$$\psi^*_{in}(R-0, \theta) = q_{jn}(\theta), \quad 0 \leqslant \theta \leqslant \omega, \quad j = 1, 2, 3, \quad n = 0, \pm 1, \pm 2, \ldots \tag{3.1}$$

It is more convenient to consider positive and negative values of the parameter n separately: $n = m$, $m > 0$ and $n = -m$, $m > 0$, and to use the following notation for the unknown discontinuities

$$\begin{aligned} &[\psi^{*\prime}_{in}(R, \tau)]_{n=m} = \chi^+_{m+1}(\tau), \quad [H'_n(R, \tau)]_{n=m} = \chi^0_m(\tau) \\ &[\psi^{*\prime}_{in}(R, \tau)]_{n=-m} = \chi^-_{m-1}(\tau), \quad [H'_n(R, \tau)]_{n=-m} = \chi^*_m(\tau), \quad m = 0, 1, 2, \ldots \end{aligned} \tag{3.2}$$

Then, using formulae (1.7) and (2.11) and the obvious linear combinations of the resulting equations, we obtain the equations

$$\begin{aligned} &I^+_{m+1} \equiv \int \sin\tau S_{m+1}(\theta, \tau)\chi^+_{m+1}(\tau)d\tau = U_m(\theta) + iq_{3,m}(\theta), \quad m = 0, 1, 2, \ldots \\ &I^-_{m-1} \equiv \int \sin\tau S_{m-1}(\theta, \tau)\chi^-_{m-1}(\tau)d\tau = U_{-m}(\theta) + iq_{3,-m}(\theta), \quad m = 0, 1, 2, \ldots \\ &I^0_m \equiv \int \sin\tau S_m(\theta, \tau)\chi^0_m(\tau)d\tau = V_m(\theta), \quad m = 0, 1, 2, \ldots \\ &I^*_m \equiv \int \sin\tau S_m(\theta, \tau)\chi^*_m(\tau)d\tau = V_{-m}(\theta), \quad m = 0, 1, 2, \ldots \end{aligned} \tag{3.3}$$

Here

$$\begin{aligned} U_n(\theta) &= \sin\theta\, q_{1,n}(\theta) + \cos\theta\, q_{2,n}(\theta) \\ V_n(\theta) &= \cos\theta\, q_{1,n}(\theta) - \sin\theta\, q_{2,n}(\theta) \end{aligned} \tag{3.4}$$

$$S_m(\theta, \tau) = \sum_{k=m}^{\infty} \frac{\sigma_{k,m}}{2k+1} P^m_k(\cos\theta) P^m_k(\cos\tau), \quad m = 0, 1, 2, \ldots \tag{3.5}$$

From the second formula of (3.4), it is clear that the solutions of the third and fourth equations of (3.3) are related by the formula $\overline{\chi^*_m(\tau)} = \chi^0_m(\tau)$. Inverting the Fourier transforms (2.11), and using (3.2) and (3.8), we find

$$H(R-0, \theta, \varphi) = R\{I^0_0 + 2\operatorname{Re} \sum_{m=1}^{\infty} I^0_m e^{im\varphi}\}$$

$$\psi^*_l(R-0, \theta, \varphi) = R\left\{\sum_{m=0}^{\infty} I^+_{m+1} e^{im\varphi} + \sum_{m=1}^{\infty} I^-_{m-1} e^{-im\varphi}\right\} \tag{3.6}$$

In order to calculate the stress intensity factor, we need to know the stress distribution on the continuation of the spherical slit, that is, $\sigma_r(R, \theta, \varphi)$, $\tau_{r\theta}(R, \theta, \varphi)$, $\tau_{r\varphi}(R, \theta, \varphi)$ for $\theta > \omega$. From (1.4), for example, $\sigma_r(R, \theta, \varphi) = R^{-1}\psi_1{}^*(R-0, \theta, \varphi)$, and from (1.7) and (3.6) we have

$$\sigma_r(R, \theta, \varphi) = \cos\theta I^0_0 + \operatorname{Re}\left\{2\cos\theta \sum_{m=1}^{\infty} e^{im\varphi} I^0_m + \right.$$

$$\left. + \sin\theta\left[\sum_{m=0}^{\infty} e^{im\varphi} I^+_{m+1} + \sum_{m=1}^{\infty} e^{im\varphi} I^+_{m+1} + \sum_{m=1}^{\infty} e^{-im\varphi} I^-_{m-1}\right]\right\}, \quad \theta > \omega \tag{3.7}$$

Once the integral equations (3.3) have been solved, this formula can be used to calculate the normal stress intensity factor, and those equations can be solved merely by finding the solution of the equation

$$\int \sin\tau\, S_m(\theta,\tau)\chi_m(\tau)d\tau = f_m(\theta),\quad 0\leqslant\theta\leqslant\omega \tag{3.8}$$

4. SOLUTION OF THE INTEGRAL EQUATION OF THE PROBLEM IN THE CLASS OF INTEGRABLE FUNCTIONS

We will attempt to sum the series (3.5), defining the kernel of the integral equation (3.8), by establishing a relation between the associated Legendre functions $P_k^m(x)$ and Jacobi polynomials $P_k^{\alpha,\beta}(x)$. Using formula 8.704 of [9], we have

$$P_k^m(x) = \frac{1}{\Gamma(1-m)}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}m} F\left(-k,\ k+1;\ 1-m;\ \frac{1-x}{2}\right) \tag{4.1}$$

Comparing this with 8.902(1) of [9] and putting $x=\cos\theta$, we obtain the relation

$$P_k^{-m,m}(\cos\theta) = [\Gamma(k+1-m)/k!]\ \mathrm{tg}^m\, \tfrac{1}{2}\theta\, P_k^m(\cos\theta) \tag{4.2}$$

Then, putting $\alpha=-m$, $\beta=m$, $t=1$, $x=\cos\theta$, $y=\cos\tau$ in formula 5.14.4(1) of [10], using (3.5) and (4.2), and then the well-known series representation of the Appel function F_4 in order to eliminate the multiplier $\Gamma(1-m)$, we obtain

$$S_m(\theta,\tau) = \frac{(\frac{1}{2})_m \sin^{2m}\frac{1}{2}\theta\,\sin^{2m}\frac{1}{2}\tau}{m!\,2\,\mathrm{tg}^m\frac{1}{2}\theta\ \mathrm{tg}^m\frac{1}{2}\tau}\ \times$$

$$\times\, F_4(\tfrac{1}{2}+m,\ 1+m;\ 1+m,\ 1+m;\ \sin^2\tfrac{1}{2}\theta\sin^2\tfrac{1}{2}\tau,\ \cos^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\tau) \tag{4.3}$$

In the reduction formula on p. 231 of [11] we take $\alpha=\frac{1}{2}+m$, $\beta=\frac{1}{2}+m$ and $x=-\mathrm{tg}^2\frac{1}{2}\theta$, $y=-\mathrm{ctg}^2\frac{1}{2}\tau$ when $\theta<\tau$ and $x=-\mathrm{tg}^2\frac{1}{2}\tau$, $y=-\mathrm{ctg}^2\frac{1}{2}\theta$ when $\tau<\theta$. The last factor in (4.2) can then be represented in the form

$$\frac{F(\frac{1}{2}+m,\ \frac{1}{2};\ 1+m;\ \mathrm{tg}^2\frac{1}{2}\theta\,\mathrm{ctg}^2\frac{1}{2}\tau)}{(\cos\frac{1}{2}\theta\,\sin\frac{1}{2}\tau)^{2m+1}},\quad \theta<\tau$$

When $\theta>\tau$, the variables θ and τ in this expression must be interchanged.

Consider the discontinuous Weber–Sonin integral

$$W_m(\mathrm{tg}\tfrac{1}{2}\theta,\ \mathrm{tg}\tfrac{1}{2}\tau) = \int_0^\infty J_m(s\,\mathrm{tg}\tfrac{1}{2}\theta)J_m(s\,\mathrm{tg}\tfrac{1}{2}\tau)ds =$$

$$= \frac{(\frac{1}{2})_m\,\mathrm{tg}^m\frac{1}{2}\theta}{m!\,\mathrm{tg}^{m+1}\frac{1}{2}\tau} F\left(m+\tfrac{1}{2},\tfrac{1}{2};\ m+1;\ \frac{\mathrm{tg}\frac{1}{2}\theta}{\mathrm{tg}\frac{1}{2}\tau}\right),\quad \theta<\tau \tag{4.4}$$

where we have used formula 6.574(1) of [9].

To obtain a representation of the integral when $\theta>\tau$, we must use formula 6.574(3) of [9]. Comparing (4.4) with the previous formula, we obtain

$$S_m(\theta,\ \tau) = \tfrac{1}{2}\sec\tfrac{1}{2}\theta\,\sec\tfrac{1}{2}\tau\, W_m(\mathrm{tg}\tfrac{1}{2}\theta,\mathrm{tg}\tfrac{1}{2}\tau),\quad m=0,1,2,\ldots \tag{4.5}$$

Taking this into account, integral equation (3.8), by means of the substitutions

$$\begin{gathered}\mathrm{tg}\tfrac{1}{2}\theta = r,\quad \mathrm{tg}\tfrac{1}{2}\tau=\rho,\quad \mathrm{tg}\tfrac{1}{2}\omega=a\\ 2(1+\rho^2)^{-3/2}\chi_m(2\,\mathrm{arctg}\,\rho) = X_m(\rho)\\ (1+r^2)^{-1/2}f_m(2\,\mathrm{arctg}\,r) = F_m(r)\end{gathered} \tag{4.6}$$

can be converted into the integral equation

$$\int_0^1 W_m(x, y) y X_m(ay) dy = \frac{1}{a} F_m(ax), \quad 0 \leqslant x \leqslant 1, \quad m = 0, 1, 2, \ldots \tag{4.7}$$

The last equation has an exact solution either in the form of quadratures or as a series in Jacobi polynomials [8]. For our present purpose, it is better to use a series solution, and this can be obtained by the method of orthogonal polynomials [8] in the form

$$X_m^0(ay) = \sum_{k=0}^{\infty} \chi_{mk} \frac{y^m}{\sqrt{1-y^2}} P_k^{m, -½}(1-2y^2) \tag{4.8}$$

$$\chi_{mk} = \frac{4(k)!(m + ½ + 2k)}{a\Gamma^2(k + ½)} \int_0^1 \frac{F_m(ax) x^{m+1} P_k^{m, -½}(1-2x^2) dx}{\sqrt{1-x^2}}$$

Substituting the resulting solutions into (3.9), we find the intensity factor by passing to the limit

$$N_1(\varphi) = \lim_{\theta \to \omega + 0} \sigma_r(R, \theta, \varphi) \sqrt{\theta - \omega} \tag{4.9}$$

However, after calculations similar to those carried out previously in [5], we come to the conclusion that $N_1(\varphi) = 0$ for any loading of the elastic medium. This shows that the solution (4.8) of (3.10) or (4.7) in the class of integrable functions does not enable the true stress distribution (3.7) to be found for $\theta = \omega$. Thus, as in [5], we need to widen the class of required solutions to include those with non-integrable singularities when $\theta = \omega$.

5. CONSTRUCTION OF THE SOLUTION OF THE NON-INTEGRABLE EQUATION IN THE CLASS OF NON-INTEGRABLE FUNCTIONS AND CALCULATION OF THE STRESS INTENSITY FACTOR

We shall construct a solution of the integral equation (4.7) in the form

$$X_m(ay) = X_m^0(ay) + C_m \frac{y^m}{(1-y^2)^{3/2}} \tag{5.1}$$

Using the same argument as in [5], we find the coefficients C_m which, in this case, will have the form

$$C_m = \frac{\chi_{m0}}{2m+1} = \frac{2}{\pi} \int_0^1 \frac{F_m(ax) x^{m+1} dx}{\sqrt{1-x^2}} \tag{5.2}$$

The solution of (4.7), given by substituting (5.1) and (4.9) into (3.7) and then using the substitution (4.6), will give a correct stress distribution (3.7) for $\theta > \omega$, allowing for the fact that only the second term in (5.1) will give an infinite root of (3.7). To make the passage to the limit (4.9), we must use the relation

$$\int \frac{\sin \tau S_l(\theta, \tau) \operatorname{tg}^l ½\tau \, d\tau}{2\cos^3 ½\tau R^3(\omega, \tau)} = \frac{\operatorname{tg}^{2l-1} ½\omega \operatorname{tg}^{-l} ½\theta}{\cos ½\theta R(\theta, \omega)},$$
$$\theta > \omega, \quad l = 0, 1, 2, \ldots, \quad R(\omega, \tau) = \sqrt{\operatorname{tg}^2 ½\omega - \operatorname{tg}^2 ½\tau} \tag{5.3}$$

which follows from (4.3) of [5], taking into account the replacement of variables (4.6). As a result of passing to the limit, we have

$$N_1(\varphi) = -\operatorname{tg}^{3/2} ½\omega \{\cos\omega \chi_{00}^0 + \operatorname{Re}[2\cos\omega \sum_{m=1}^{\infty} \frac{l^{im\varphi} \chi_{m0}^0}{2m+1} +$$
$$+ \sin\omega (\sum_{l=1}^{\infty} \frac{e^{i(l-1)\varphi}}{2l+1} \chi_{l0}^+ + \sum_{l=0}^{\infty} \frac{e^{-i(l+1)\varphi}}{2l+1} \chi_{l0}^-)]\} \tag{5.4}$$

By (5.2), (4.6), (3.3) and (3.4) we have

$$\chi^0_{m0} = \frac{2m+1}{\pi\,\mathrm{tg}\tfrac{1}{2}\omega}\int_0^\omega \frac{V_m(\theta)C^{m+1}(\theta)d\theta}{\cos\tfrac{1}{2}\theta\, R(\omega,\theta)}; \qquad C(\theta) = \frac{\mathrm{tg}\tfrac{1}{2}\theta}{\mathrm{tg}\tfrac{1}{2}\omega}, \qquad m = 0, 1, 2, \ldots$$

$$\chi^+_{l0} = \frac{2l+1}{\pi\,\mathrm{tg}\tfrac{1}{2}\omega}\int_0^\omega \frac{U_{l-1}(\theta)+iq_{3,l-1}(\theta)}{\cos\tfrac{1}{2}\theta\, R(\omega,\theta)}C^{l+1}(\theta)d\theta, \quad l = 1, 2, 3, \ldots \tag{5.5}$$

$$\chi^-_{l0} = \frac{2l+1}{\pi\,\mathrm{tg}\tfrac{1}{2}\omega}\int_0^\omega \frac{U_{-(l+1)}(\theta)+iq_{3,-(l+1)}(\theta)}{\cos\tfrac{1}{2}\theta\, R(\omega,\theta)}C^{l+1}(\theta)d\theta, \quad l = 0, 1, 2, \ldots$$

We will transform the resulting formula (5.4) into a more convenient form. Substituting (5.5) into (5.4) and writing out the real part of the complex expression in (5.4) in the form of a sum of conjugates, we obtain

$$N_1(\varphi) = -\frac{\mathrm{tg}^{1/2}\tfrac{1}{2}\omega}{\pi}\int_0^\omega \frac{\Omega(\theta,\varphi)d\theta}{\cos\tfrac{1}{2}\theta\, R(\omega,\theta)},$$

$$\Omega(\theta,\varphi) = \cos\omega\, C(\theta)\sum_{m=-\infty}^{m=\infty} e^{im\varphi}V_m(\theta)C^m(\theta) +$$

$$+\tfrac{1}{2}\sin\omega\,\{C^2(\theta)[U_0(\theta) + \sum_{m=-\infty}^{\infty} e^{im\varphi}U_m(\theta)C^{|m|}(\theta) + \tag{5.6}$$

$$+\sum_{m=-\infty}^{\infty} e^{im\varphi}U_m(\theta)C^{|m|}(\theta) - U_0(\theta)\}$$

Using the theorem of a convolution for a finite Fourier transform (see [8], p. 316, for instance) and formula 1.447(3) [9], taking account of (3.4) we have

$$N_1(\varphi) = -\frac{\mathrm{tg}^{1/2}\tfrac{1}{2}\omega}{2\pi^2}\int_0^\omega\int_{-\pi}^{\pi} \frac{C(\theta)[1-C^2(\theta)][\cos\omega V(\theta,\psi)+\sin\omega U(\theta,\psi)\cos(\varphi-\psi)]d\theta d\psi}{\cos\tfrac{1}{2}\theta\, R(\omega,\theta)C(\theta,\varphi-\psi)} \tag{5.7}$$

where

$$U(\theta,\psi) = \sin\theta\, q_1(\theta,\psi) + \cos\theta q_2(\theta,\psi)$$
$$V(\theta,\psi) = \cos\theta q_1(\theta,\psi) - \sin\theta q_2(\theta,\psi) \tag{5.8}$$
$$C(\theta,\Phi) = 1 - 2C(\theta)\cos\Phi + C^2(\theta)$$

We also obtain formulae for the shear stress intensity factors

$$N_2(\varphi) = \lim_{\theta\to\omega+0} \tau_{r\theta}(R,\theta,\varphi)\sqrt{\theta-\omega}$$
$$N_3(\varphi) = \lim_{\theta\to\omega+0} \tau_{r\varphi}(R,\theta,\varphi)\sqrt{\theta-\omega} \tag{5.9}$$

According to (1.4), the stresses on the continuation of the crack that are included here will be expressed as

$$\tau_{r\theta}(R,\theta,\varphi) = R^{-1}\psi_2^*(R-0,\theta,\varphi)$$
$$\tau_{r\theta}(R,\theta,\varphi) = R^{-1}\psi_3^*(R-0,\theta,\varphi), \quad \theta>\omega \tag{5.10}$$

The use of formulae (1.7) and (3.6) and calculations similar to those carried out to obtain (5.7) lead to the result

$$N_2(\varphi) = -\frac{\mathrm{tg}^{\frac{1}{2}}\,\frac{1}{2}\omega}{2\pi^2}\int_0^{\omega}\int_{-\pi}^{\pi}\frac{C(\theta)[1-C^2(\theta)]\,[\cos\omega U(\theta,\psi)\cos(\varphi-\psi)-\sin\omega\, V(\theta,\psi)]d\theta d\psi}{\cos\frac{1}{2}\theta R(\omega,\theta)\; C(\theta,\varphi-\psi)}$$

$$N_3(\varphi) = -\frac{\mathrm{tg}^{\frac{1}{2}}\,\frac{1}{2}\omega}{2\pi^2}\int_0^{\omega}\int_{-\pi}^{\pi}\frac{C(\theta)[1-C^2(\theta)]\cos(\varphi-\psi)q_3(\theta,\psi)d\theta d\psi}{\cos\frac{1}{2}\theta R(\omega,\theta)C(\theta,\varphi-\psi)} \tag{5.11}$$

6. SOME SPECIAL CASES OF THE LOADING OF AN ELASTIC MEDIUM

We will start with the case of all-round extension of the elastic medium at infinity by a uniform load of intensity p. In this case

$$q_1(\theta,\varphi) = -p, \quad q_2(\theta,\varphi) = 0, \quad q_3(\theta,\varphi) = 0$$

and therefore, according to (5.8)

$$U(\theta,\varphi) = -p\sin\theta, \quad V(\theta,\varphi) = -p\cos\theta \tag{6.1}$$

Thus, in this case $N_3(\varphi) = 0$, and for $N_1(\varphi) = N_1$, $N_2(\varphi) = N_2$ according to (5.7) and (5.11) and from formulae 3.613 of [9], we have the expressions

$$N_1 = \frac{1}{\pi\,\mathrm{tg}^{\frac{1}{2}}\frac{1}{2}\omega}\left[\cos\omega J^*(\omega) + \frac{2\sin\omega J(\omega)}{\mathrm{tg}\,\frac{1}{2}\omega}\right] \tag{6.2}$$

$$N_2 = \frac{1}{\pi\,\mathrm{tg}^{\frac{1}{2}}\frac{1}{2}\omega}\left[\frac{2\cos\omega J(\omega)}{\mathrm{tg}\frac{1}{2}\omega} - \sin\omega J^*(\omega)\right]$$

where

$$J(\omega) = \int\frac{\mathrm{tg}^2\frac{1}{2}\theta\sin\frac{1}{2}\theta\, d\theta}{R(\omega,\theta)}, \quad J^*(\omega) = \int\frac{\mathrm{tg}\frac{1}{2}\theta\cos\theta\, d\theta}{\cos\frac{1}{2}\theta R(\omega,\theta)} \tag{6.3}$$

and, from the relation $\cos\theta = \cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta$, the last integral reduces to

$$J^*(\omega) = \int\frac{\sin\frac{1}{2}\theta\, d\theta}{R(\omega,\theta)} - J(\omega)$$

The integrals here can be reduced to standard tabulated form by the substitution $\theta = 2\,\mathrm{arctg}\,x$. We then have

$$J(\omega) = \omega - \sin\omega, \quad J^*(\omega) = 2\sin\omega - \omega \tag{6.4}$$

and, therefore, instead of (6.2) we will have the expressions

$$N_1 = \frac{p[\omega\cos\omega + 2(\omega - \sin\omega)]}{\pi\,\mathrm{tg}\,\frac{1}{2}\omega}$$

$$N_2 = \frac{2}{\pi}\,\frac{p\cos\frac{1}{2}\omega}{\mathrm{tg}^{3/2}\frac{1}{2}\omega}\,(\omega\cos\tfrac{1}{2}\omega - 2\sin\tfrac{1}{2}\omega)$$

The next case that we consider is that of axial extension at infinity in the direction ν by a uniform load of intensity p. We introduce the following notation for the direction cosines

$$\cos(\nu, x) = l_p, \quad \cos(\nu, y) = m_p, \quad \cos(\nu, z) = n_p \tag{6.5}$$

Using well-known formulae to transform the components of the stress tensor by rotating the coordinate axes, in this case, we have

$$q_1(\theta,\varphi) = -p[\sin^2\theta(l_p^2\cos\varphi + m_p^2\sin^2\varphi + l_pm_p\sin 2\varphi) + n_p\sin 2\theta(l_p\cos\varphi + m_p\sin\varphi)]$$

$$2q_2(\theta,\varphi) = -p[\sin 2\theta(l_p^2\cos^2\varphi + m_p^2\sin^2\varphi - n^2{}_p + l_pm_p\sin 2\varphi) + 2l_pn_p\cos 2\theta\cos\varphi +$$

$$+ 2m_pn_p(\cos^2\theta\cos\varphi - \sin^2\theta\sin\varphi)]$$

$$2q_3(\theta,\varphi) = -p[\sin 2\varphi(m_p^2\sin\theta - l_p^2\cos\theta) + l_pm_p\sin\theta\cos 2\varphi + n_p\cos\theta(m_p - l_p\sin\varphi)] \tag{6.6}$$

and, according to (5.8)

$$U(\theta,\varphi) = -p[\tfrac{1}{2}(l_p^2 + m_p^2)\sin\theta - n_p^2 \sin\theta\cos^2\theta + \tfrac{1}{2}(l_p^2 - m_p^2)\sin\theta\cos 2\varphi + l_p m_p \sin\theta \sin 2\varphi +$$
$$+ n_p(l_p\cos\theta + m_p\cos^3\theta) + n_p m_p \sin^2\theta\cos\theta\sin\varphi],$$
$$V(\theta,\varphi) = -p[n_p^2\sin^2\theta\cos\theta + n_p\sin\theta(l_p - m_p\cos^2\theta)\cos\varphi + n_p m_p \sin\theta\cos^2\theta\sin\theta] \tag{6.7}$$

We now substitute these expressions into (5.7). Subsequent use of integrals 3.613 [9] yields a single quadrature for the intensity factor $N_1(\varphi)$, where the integrals have the same structure as in (6.3), and the same substitution gives functions tabulated in [9]. Thus, we can obtain

$$2\pi p^{-1}\,\mathrm{tg}\tfrac{1}{2}\omega N_1(\varphi) = 2n_p^2 J_1(\omega)\cos\omega + \sin\omega\,\mathrm{ctg}\tfrac{1}{2}\omega[2(l_p^2 + m_p^2 - n_p^2)J(\omega) +$$
$$+ n_p^2 J_3(\omega)] + n_p m_p\,\mathrm{ctg}\tfrac{1}{2}\omega[2J(\omega) - J_3(\omega)]\cos\omega\sin\varphi + n_p m_p\sin\omega[J_1(\omega) +$$
$$+ \mathrm{ctg}^2\tfrac{1}{2}\omega\, J_5(\omega)](\sin\varphi - \cos\varphi) + \{2n_p\cos\omega\,\mathrm{ctg}\tfrac{1}{2}\omega[2(l_p - m_p)J(\omega) +$$
$$+ m_p J_3(\omega)] + n_p(l_p + m_p)\sin\omega[J^*(\omega) + \mathrm{ctg}^2\tfrac{1}{2}\omega J_1^*(\omega)]\}\cos\varphi + [J(\omega) + \mathrm{ctg}^2\tfrac{1}{2}\omega J_4(\omega)] \times$$
$$\times \sin\omega\,\mathrm{ctg}\tfrac{1}{2}\omega[(l_p^2 - m_p^2)\cos 2\varphi + 2l_p m_p\sin 2\varphi] \tag{6.8}$$

The integrals $J(\omega)$ and $J^*(w)$ are given by formulae (6.4), and for the others we have

$$J_1(\omega) = \int \sin^2\tau\, Q_\omega^{(1)}(\tau)\,d\tau = 8\,\mathrm{tg}^3\tfrac{1}{2}\omega\,[\cos\omega(\cos^4\tfrac{1}{2}\omega + \tfrac{2}{3}\sin^2\tfrac{1}{2}\omega + \cos^2\tfrac{1}{2}\omega + \tfrac{1}{5}\sin^4\tfrac{1}{2}\omega) +$$
$$+ (\sin^2\tfrac{1}{2}\omega - \cos\omega)(\tfrac{1}{3}\cos^2\tfrac{1}{2}\omega + \tfrac{1}{5}\sin^2\tfrac{1}{2}\omega) - \tfrac{1}{5}\sin^2\tfrac{1}{2}\omega]$$
$$Q_\omega^{(k)}(\tau) = \mathrm{tg}^k\tfrac{1}{2}\tau\cos\tau\sec\tfrac{1}{2}\tau R^{-1}(\omega,\tau), \qquad k = 1, 2, 3, 4$$
$$J_1^*(\omega) = \int Q_\omega^{(3)}(\tau)\,d\tau = 2\,\mathrm{tg}\tfrac{1}{2}\omega(\mathrm{tg}^2\tfrac{1}{2}\omega\cos\omega + \mathrm{tg}^2\tfrac{1}{2}\omega - 2 - \tfrac{1}{2}\mathrm{tg}\tfrac{1}{2}\omega) - \tfrac{1}{2}\omega(\mathrm{tg}^2\tfrac{1}{2}\omega - 5)$$
$$J_3(\omega) = \tfrac{1}{2}\int \sin^2\tau\,\mathrm{tg}\tau\, Q_\omega^{(2)}(\tau)\,d\tau = 16\,\mathrm{tg}^3\tfrac{1}{2}\omega\sin^2\tfrac{1}{2}\omega(\cos^4\tfrac{1}{2}\omega + \tfrac{2}{3}\cos^2\tfrac{1}{2}\omega + \tfrac{1}{5}\sin^4\tfrac{1}{2}\omega -$$
$$- \cos^2\tfrac{1}{2}\omega - \tfrac{2}{3}\sin^2\tfrac{1}{2}\omega + \tfrac{1}{5})$$
$$J_4(\omega) = \tfrac{1}{2}\int \mathrm{tg}\tau\, Q_\omega^{(4)}(\tau)\,d\tau = 2\,[\mathrm{tg}\tfrac{1}{2}\omega(\sin^2\tfrac{1}{2}\omega - 2\cos^2\tfrac{1}{2}\omega + \omega\,\mathrm{tg}\tfrac{1}{2}\omega + \tfrac{1}{2} + \cos^{-2}\tfrac{1}{2}\omega - \tfrac{3}{4}\omega]$$
$$J_5(\omega) = \tfrac{1}{2}\int \sin\tau\sin 2\tau\, Q_\omega^{(3)}(\tau)\,d\tau = 8\,\mathrm{tg}^3\tfrac{1}{2}\omega\{\cos\omega(\cos^4\tfrac{1}{2}\omega + \tfrac{2}{3}\sin^2\tfrac{1}{2}\omega\cos^2\tfrac{1}{2}\omega + \tfrac{1}{5}\sin^4\tfrac{1}{2}\omega) +$$
$$+ \mathrm{tg}^2\tfrac{1}{2}\omega[(\sin^2\tfrac{1}{2}\omega - 2\cos\omega)(\tfrac{1}{3}\cos^2\tfrac{1}{2}\omega + \tfrac{1}{5}\sin^2\tfrac{1}{2}\omega) - \tfrac{1}{5}(2\cos\omega - 1)]\} +$$
$$+ \tfrac{23}{15}\mathrm{tg}^5\tfrac{1}{2}\omega - \tfrac{7}{3}\mathrm{tg}^3\tfrac{1}{2}\omega\sin^{-2}\tfrac{1}{2}\omega + \mathrm{tg}\tfrac{1}{2}\omega\sin^{-4}\tfrac{1}{2}\omega - \tfrac{1}{2}\omega$$

Thus, for the given loading, the normal stress intensity factor can be expressed in terms of elementary functions. This is also true of the shear stress intensity factor (5.9). For, substituting (6.7) into (5.11), the formulae obtained for them are similar to (6.8).

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Translated by R.L.